

Spatial Resolution Properties of Mapped Spectral Chebyshev Methods

Bruno Costa *

Wai-Sun Don †

Aline Simas ‡

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Abstract

In this article we clarify some fundamental questions on the resolution power of a modified spectral Chebyshev method proposed by Kosloff and Tal-Ezer and state approximation properties of general mappings of Chebyshev points. We develop a technique based on the method of the stationary phase which provides a straightforward way to determine the spatial resolution power of the Chebyshev method with mappings. In particular, we prove a conjecture about the resolution power of the Kosloff–Tal-Ezer mapping, yielding, as a corollary, a rigorous demonstration that a minimum of π polynomials per wavelength are needed in the original Chebyshev method, a well known fact which has been only heuristically demonstrated in the past literature.

Keywords

Chebyshev Collocation, Kosloff and Tal-Ezer Map, Resolution

AMS

65M70, 65M12, 65P30, 77Axx

1 Introduction

The nonuniform distribution of the Chebyshev collocation points

$$\xi_j = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, \dots, N, \quad (1)$$

leads to tight temporal stability restrictions when solving Partial Differential Equations with Spectral Chebyshev collocation methods. Hyperbolic equations have a time step restriction of $\Delta t = O(N^{-2})$ for explicit temporal integration formulae, while for finite difference scheme they are $\Delta t = O(N^{-1})$, where N is the number of grid points. The source of such a restrictive time step is credited to the agglomeration of the Chebyshev points near the boundaries. The minimal grid spacing of order $\Delta x_{\min} = O(N^{-2})$ yields a spectral radius of size $O(N^2)$ for the first derivative matrix [9, 12]. It can be argued that smaller grid spacing resolves higher wavenumbers and, therefore, faster dynamics, requiring shorter time steps.

*Departamento de Matemática Aplicada, IM-UFRJ, Caixa Postal 68530, Rio de Janeiro, RJ, C.E.P. 21945-970, Brazil. E-Mail: bcosta@labma.ufrj.br

†Brown University, Division of Applied Mathematics, Box F, 182 George Street, Providence, RI 02912, USA. E-Mail: wson@cfm.brown.edu

‡Departamento de Matemática Aplicada, IM-UFRJ, Caixa Postal 68530, Rio de Janeiro, RJ, C.E.P. 21945-970, Brazil. E-Mail: bcosta@labma.ufrj.br

Kosloff and Tal-Ezer [10] have addressed this issue and proposed to map the Chebyshev points to a new set of collocation points with a greater minimal grid spacing in order to alleviate the time step restriction. We will hereafter refer to this modified method as the mapped Chebyshev method. The mapping is given by

$$x = g(\xi, \alpha) = \frac{\arcsin(\alpha\xi)}{\arcsin(\alpha)}, \quad (2)$$

and $\alpha \in (0, 1)$ is a parameter determining the strength of the endpoints separation. The proposed mapping stretches the grid spacing on the boundaries, pushing the points to the center of the domain, generating a quasi-uniform grid. In their seminal work, they claimed that the mapping decreases the spectral radius of the derivative operator from $O(N^2)$ to $O(N)$, increasing the allowed time step from $O(N^{-2})$ to $O(N^{-1})$ for hyperbolic problems with some specific choice of α .

While many good properties arise from this new distribution of points, smaller roundoff error and increased resolution for higher modes, the map has branch point singularities at $y = \pm\alpha^{-1}$, introducing an interpolation error. In [10], a way of choosing the parameter α in order to avoid the influence of these singularities in the numerical scheme was proposed and in [1], numerical results confirmed the effectiveness of the proposed numerical fix.

In [10] a conjecture about the resolution power of the new set of points (2) stated a better spatial resolution when compared to the classical Chebyshev points (1) and confirmed numerically [1]. However, only numerical results involving the decay of the spectral coefficients were shown. In this work, we investigate more carefully the decay of the spectral coefficients and through the analysis of the oscillatory integrals defining them, were able to come up with an analytical proof for the conjecture. The technique developed along with the proof provides a simple test to measure the resolution power of general mappings of the Chebyshev points. Particularly, the test applied to the Chebyshev points confirms the well-known result of π points per wavelength for spatial resolution, a fact which has been only indirectly demonstrated in the literature through an association with the decay of spectral Bessel coefficients on a Chebyshev series [5].

This article is divided as follows: In Section 2, we present the mapped method and a discussion on the choice of the parameter α with regards to the resolution results already known in the literature. In Section 3, we present an heuristic discussion on the spatial resolution of Chebyshev and mapped grids prior to presenting the main result of this article, which is the proof of the conjecture on the spatial resolution power of the mapped Chebyshev method. Finally, in Section 4, we extend the theory to general mappings of Chebyshev points, applying the results to draw conclusions about the resolution power of equi-spaced points and of the classical set of Chebyshev collocation points.

2 Mapped Chebyshev collocation methods

The Kosloff–Tal-Ezer mapping (2) stretches those Chebyshev collocation points (1) that are close to the boundary and pushes the interior points towards the center of the interval $[-1, 1]$. This fact has two important consequences on the numerical properties of the mapped Chebyshev method. First, the maximum time step $\Delta t = O(\Delta x_{\min})$ is increased and, secondly, the maximum grid spacing Δx_{\max} is diminished, increasing resolution of the higher modes [1, 6, 10]. Moreover, we have

- $g'(\xi) = \frac{\alpha}{\arcsin(\alpha)} \frac{1}{\sqrt{1-(\alpha\xi)^2}}$ is singular at $\xi = \pm\alpha^{-1}$;
- $\Delta x_{\min} \rightarrow \frac{2}{N}$ if $\alpha \rightarrow 1$;
- $\Delta x_{\min} \rightarrow 1 - \cos(\frac{\pi}{N})$ if $\alpha \rightarrow 0$;

- $\Delta x_{\min} = \frac{2}{\pi N} \left(\sqrt{\pi^2 + 2c} - \sqrt{2c} \right)$ if $\alpha = 1 - cN^{-2} + O(N^{-3}), c > 0$.

It can be seen that the time step Δt is increased from $\Delta t = O(N^{-2})$ to $\Delta t = O(N^{-1})$ as α increases from 0 to 1; however, the singularities of the mapping introduce an approximation error ε in the L^∞ -norm. According to [7], the following relation exists between α, N and ε :

$$\alpha = \operatorname{sech} \left(\frac{|\ln \varepsilon|}{N} \right). \quad (3)$$

From (3), we can infer a give and take situation between accuracy and resolution for a grid with a fixed number of points N . Decreasing α , decreases the mapping error ε , increasing the overall accuracy. However, a smaller α means a larger Δx_{\max} at the center of the interval, sacrificing spatial resolution for higher modes [10]. If the primary interest is accuracy, one should set ε as the machine zero ε_M , avoiding in this way any influence of the mapping error in the computations.

In some applications, choosing $\varepsilon = \varepsilon_M$ might be too stringent and a better alternative may be to set ε equal to the desired approximation error and allow the use of a bigger Δt . In [6], a value of $\alpha = \cos(\frac{1}{2})$ enabled the use of a time step two times larger when applying the mapping to a multi-domain simulation of the Maxwell's equations, with a value of ε about 10^{-7} (a single precision setting). Nevertheless, It should be noticed that this choice of α only allowed the spatial resolution of the lower 2/3 Fourier spectrum to the given accuracy of $\varepsilon = 10^{-7}$, while leaving the upper 1/3 Fourier spectrum unresolved. In effect, the increase of the time step Δt was obtained at the price of solving for less than the optimal number of Fourier modes that could be solved. For applications which are non-linear in nature, the use of a smaller N , with the choice of $\varepsilon = \varepsilon_M$, is a wiser approach, since the generation of unresolved high modes demands an increase in the number of grid points.

3 Spatial Resolution Power of the Mapped Chebyshev Method

More clear notions of what is meant in this work by spatial resolution and accuracy are necessary for the discussion that follows. While in the context of the classical Finite Differences and Finite Elements methods there is a clear separation between these concepts, their distinction in the case of Spectral methods is more subtle. Spatial resolution is achieved when the truncation error starts to decrease. Accuracy is defined as the order of the rate of decrease. In spectral expansions, particularly the ones of singular Sturm-Liouville problems, exponential decay is guaranteed by the smoothness of the expanded function. Spatial resolution, seen as the necessary minimum number of basis elements in the truncated series to represent a function, is the only factor preventing the exponential accuracy to take place. Therefore, in such spectral expansions, achieving spatial resolution for a specific function coincides with the beginning of the exponential decay of its spectral coefficients (see Figure (1)).

In this section we study the spatial resolution of the mapped Chebyshev method in two stages: First, we perform an heuristic comparison of the grids of collocation points of the classical and mapped Chebyshev methods to recover a conjecture stated in [10] on the spatial resolution power of the mapped Chebyshev method. Secondly, we use the method of stationary phase to prove the conjecture by deriving a relation between the degree of the truncated series and the wavenumber $\frac{m}{\pi}$ of a sinusoidal function like $\cos(mx)$ that guarantees a decrease rate of the spectral coefficients of order at least k^{-1} . This order of decay is the behavior that we consider in this article as the onset of spatial resolution and is the subject of Theorem 1 below.

3.1 The Spatial Resolution Conjecture

It is well known that on a uniformly spaced grid, the maximal resolved mode w_{\max} is the inverse of the maximum grid spacing Δx_{\max} . For instance, considering an uniformly spaced grid in $[-1, 1]$, the direct use of the Fourier Transform gives us that $\Delta x_{\max} = \frac{2}{N}$ and $w_{\max} = \frac{N}{2}$, yielding the classical result of 2 points per wave number for the spatial resolution of an equi-spaced points grid. In the case of the nonuniform Chebyshev grid (1), we consider the following expansion of a trigonometric function in terms of Chebyshev polynomials, with the aid of the Bessel functions J_n [5]:

$$\sin(mx) = 2 \sum_{n=0}^{\infty} \frac{1}{c_n} J_n(m) \sin\left(\frac{1}{2}n\pi\right) T_n(x). \quad (4)$$

Observe that at least π polynomials per wavenumber are needed in order to achieve exponential convergence of the series in (4). This is due to the facts that $J_n(m)$ goes to zero exponentially fast once n is larger than m and $\sin(mx)$ has $\frac{m}{\pi}$ wavelengths inside the interval $[-1, 1]$. This analysis, along with the maximal spacing analysis for the Chebyshev grid: $\Delta x_{\max} = \frac{\pi}{N}$ implies $w_{\max} = \frac{1}{\Delta x_{\max}} = \frac{N}{\pi}$, have been the classical heuristic arguments in the literature to confirm that the Chebyshev grid has a resolution power of π points per wave number. In Section 4 we make use of Theorem 1 below to provide a more straightforward demonstration of this fact.

What about the mapped Chebyshev method? As α goes to 1, the mapped Chebyshev grid becomes almost equally spaced and the Chebyshev polynomials become cosine functions with rational wavenumbers [10]. A maximal spacing analysis reveals that the mapping (2) stretches the Chebyshev grid away from the boundary points, forcing Δx_{\max} to decrease, improving the overall spatial resolution. Since in the Chebyshev case the maximum spacing is located at the middle of the interval $[-1, 1]$, the maximum grid spacing of the mapped Chebyshev grid is given by $\Delta x_{\max}^g = \Delta x_{\max} g'(0, \alpha) = \frac{\pi}{N} \frac{\alpha}{\arcsin(\alpha)}$. By this same maximal spacing heuristics, the maximum wave number w_{\max}^g for the mapped Chebyshev grid is given by $w_{\max}^g = \frac{N}{\pi} \frac{\arcsin(\alpha)}{\alpha} > \frac{N}{\pi}$. Therefore, only $\pi \frac{\alpha}{\arcsin(\alpha)} (< \pi)$ points per wavelength are needed for spatial resolution.

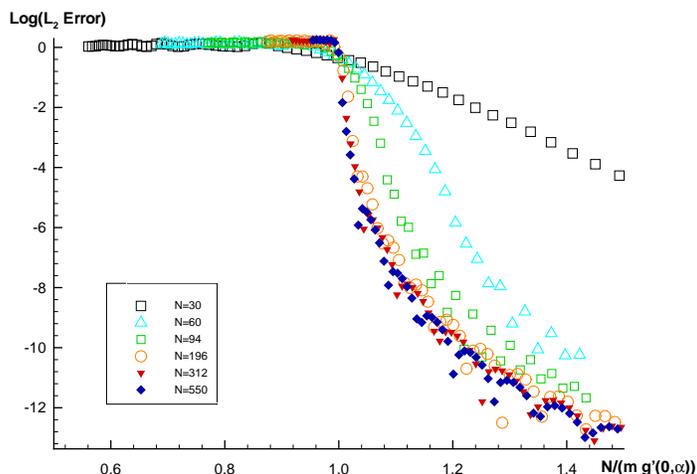


Figure 1: The normalized L_2 Error of the interpolation of the function $\cos(mx)$ by the mapped Chebyshev collocation method for various wave numbers m and numbers of collocation points N .

This result was stated as a conjecture in [10] and it has been confirmed in later works through

numerical experiments; a clear and analytical proof has never been shown. We provide such a proof in the following theorem, where a general technique providing a simple path for the computation of the spatial resolution power of general mapped Chebyshev methods is developed.

Figure 1 shows the interpolation error of the mapped Chebyshev method for the function $u(x) = \cos(mx)$ for several values of m and N . The horizontal axis is the ratio between the maximum wave number w_{\max}^g and the wavenumber $\frac{m}{\pi}$ given by

$$\frac{w_{\max}^g \pi}{m} = \frac{N \arcsin(\alpha)}{m \alpha} = \frac{N}{m g'(0, \alpha)}.$$

Note that spatial resolution is attained only when this ratio exceeds one, confirming the heuristic arguments just given.

3.2 The Spatial Resolution Theorem

In the theorem that follows we consider the Chebyshev expansion of $\cos(mx)$:

$$\cos(mx) = \sum_{k=0}^{\infty} \hat{a}_k T_k(\xi), \quad (5)$$

with x as in (2), and analyze the rate of decay of the coefficients \hat{a}_k .

The formal rate of convergence of the sequence $\{\hat{a}_k\}$, given by the behavior of the terms when as $k \rightarrow \infty$, fails to indicate the number of modes that should be retained in a finite expansion in order that the truncation error reflects the rate of convergence of the infinity expansion. Thus, it makes sense to talk about local rates of convergence for a infinite series expansion that will, in practice, be truncated. This theorem below address the issue of the minimum number of modes necessary for spatial resolution with the Kosloff–Tal-Ezer Chebyshev expansion (5) and is shown to be smaller than the one of the classical Chebyshev expansion.

Theorem 1. *A minimum of $\frac{\pi \alpha}{\arcsin(\alpha)}$ modes per wavenumber is necessary in order for the coefficients of the mapped Chebyshev expansion to start decreasing at order k^{-1} .*

Proof. The Chebyshev coefficients \hat{a}_k are given by

$$\hat{a}_k = \frac{2}{\pi} \int_{-1}^1 \frac{\cos(mx) T_k(\xi)}{\sqrt{1 - \xi^2}} d\xi = \frac{2}{\pi} \int_0^{\pi} \cos(mg(\cos(\theta), \alpha)) \cos(k\theta) d\theta, \quad (6)$$

where $\theta = \arccos(\xi)$. If k is even (odd), $\cos(k\theta)$ is even (odd) with respect to $\theta = \frac{\pi}{2}$ and $\cos(mg(\cos(\theta), \alpha))$ is always even with respect to $\frac{\pi}{2}$. Therefore, (6) is equivalent to:

$$\hat{a}_k = \begin{cases} 0, & k \text{ odd,} \\ \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos(mg(\cos(\theta), \alpha)) \cos(k\theta) d\theta, & k \text{ even.} \end{cases} \quad (7)$$

With the aid of the trigonometric identity $\cos(a) \cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$ and $\cos(a) = \text{Re}\{\exp(ia)\}$, we may rewrite (7) as

$$\hat{a}_k = \frac{2}{\pi} \text{Re}\{H^+(\theta, k, m, \alpha) + H^-(\theta, k, m, \alpha)\}, \quad (8)$$

where

$$H^\pm(\theta, k, m, \alpha) = \int_0^{\frac{\pi}{2}} \exp(i(mg(\cos(\theta), \alpha) \pm k\theta))d\theta \quad (9)$$

$$= \int_0^{\frac{\pi}{2}} \exp(ik(\beta g(\cos(\theta), \alpha) \pm \theta))d\theta, \quad (10)$$

and $\beta = \frac{m}{k}$ for $k = 0, 2, \dots, \infty$.

We are now ready to show that the local convergence of H^\pm to zero is less than k^{-1} when k is smaller than $\frac{m\alpha}{\arcsin(\alpha)}$.

The phase functions of the integrand in (10) are

$$\Psi_\pm(\theta) = \beta g(\cos(\theta), \alpha) \pm \theta. \quad (11)$$

The Riemann -Lebesgue Lemma requires that $\Psi'_\pm(\theta) = \beta g'(\cos(\theta), \alpha) \pm 1$ do not vanish in $[0, \frac{\pi}{2}]$, if \hat{a}_k is to tend to zero at least as fast as k^{-1} . Thus, we need to verify for which value(s) of θ , $\Psi'_\pm(\theta) \neq 0$ in $[0, \frac{\pi}{2}]$.

Since $g'(\cos(\theta), \alpha)$ is a decreasing function in $[0, \frac{\pi}{2}]$:

$$g'(\cos(\theta), \alpha) = \frac{-\alpha}{\arcsin(\alpha)} \frac{\sin(\theta)}{\sqrt{1 - (\alpha \cos(\theta))^2}}, \quad (12)$$

with

$$g''(\cos(\theta), \alpha) = \frac{-\alpha(1 - \alpha^2)}{\arcsin(\alpha)} \frac{\cos(\theta)}{(1 - (\alpha \cos(\theta))^2)^{\frac{3}{2}}} \leq 0, \quad (13)$$

and $g'(\cos(0), \alpha) = 0, g'(\cos(\frac{\pi}{2}), \alpha) = \frac{-\alpha}{\arcsin(\alpha)} < 0$, for $\alpha \in (0, 1)$, we obtain $g'(\cos(\theta), \alpha) \in [\frac{-\alpha}{\arcsin(\alpha)}, 0]$. Therefore

- $\Psi'_-(\theta) = \beta g'(\cos(\theta), \alpha) - 1 \leq -1 < 0$, for $\theta \in [0, \frac{\pi}{2}]$ and $\alpha \in (0, 1)$.
- $\Psi'_+(\theta)$ is bounded away from zero in $[0, \frac{\pi}{2}]$ iff $\beta |g'(\cos(\frac{\pi}{2}), \alpha)| < 1$, i.e., iff

$$k > m |g'(0, \alpha)| = m \frac{\alpha}{\arcsin(\alpha)}. \quad (14)$$

By the method of stationary phase [11], an oscillatory integral like H^+ in (10) fails to converge to zero as $O(k^{-1})$ if the phase function $\Psi_+(\theta)$ has a point where its derivative vanishes, a stationary point, and the rate of convergence is given by the order of the first non-vanishing derivative of $\Psi_+(\theta)$ at the stationary point. Applying this stationary phase argument is not straightforward in our case for $\Psi_+(\theta)$ depends on k as well, making the stationary point to vary with k .

To fix that, we define from (10) the auxiliary double sequence on k and k^* :

$$H^*(\theta, k, k^*, m, \alpha) = \int_0^{\frac{\pi}{2}} \exp\left(ik\left(\frac{m}{k^*}g(\cos(\theta), \alpha) + \theta\right)\right) d\theta. \quad (15)$$

Thus, for a fixed $k^* > m \frac{\alpha}{\arcsin(\alpha)}$, the auxiliary phase function $\Psi_*(\theta) = \frac{m}{k^*}g(\cos(\theta), \alpha) + \theta$ has no stationary point and $H^*(\theta, k, k^*, m, \alpha)$ locally converges to zero at $O(k^{-1})$. On the other hand, if

$k^* = m \frac{\alpha}{\arcsin(\alpha)}$, then $\theta = \frac{\pi}{2}$ is a stationary point for Ψ_* and $H^*(\theta, k, k^*, m, \alpha)$ locally converges to zero as $O(k^{-\frac{1}{3}})$, since

$$\Psi_*'''(\theta) = \frac{m}{k^*} \left[\frac{-\alpha(1-\alpha^2)}{\arcsin(\alpha)} \right] \left[-\sin \theta (1 - (\alpha \cos \theta)^2)^{-\frac{3}{2}} + \alpha \cos^2 \theta \sin \theta (1 - (\alpha \cos \theta)^2)^{-\frac{5}{2}} \right],$$

yielding $\Psi_*'(\frac{\pi}{2}) = \Psi_*''(\frac{\pi}{2}) = 0$ and $\Psi_*'''(\frac{\pi}{2}) \neq 0$. This show that the sequence $H^+(\theta, k, m, \alpha) = H^*(\theta, k, k, m, \alpha)$ has two distinct local rates of convergence for each fixed m . The faster decreasing order of $O(k^{-1})$ is only achieved in the range $k > m \frac{\alpha}{\arcsin(\alpha)}$. \square

We end this section pointing out that once condition (14) is satisfied, $\Psi_{\pm}(\theta)$ are monotone functions in $[0, \pi]$ and their inverse functions are well defined.

4 General Mappings

The use of the auxiliary functions (9) and (11) in the proof of Theorem 1 is not particular to the Kosloff–Tal-Ezer mapping. Only a few properties of the mapping $g(y, \alpha)$ are necessary to validate Theorem 1. The Kosloff–Tal-Ezer mapping is not unique at increasing the overall spatial resolution without destroying exponential convergence. In this section, we discuss which properties of general mappings of the Chebyshev points are necessary to obtain the same numerical improvements as the Kosloff–Tal-Ezer mapping.

As defined in Section 3, spatial resolution is attained when the coefficients of the mapped Chebyshev method, \hat{a}_k , decay at least at order $O(k^{-1})$. Note that the mapping $\tilde{g}(\theta) = g(\cos(\theta), \alpha)$ can be treated as a mapping from the equi-spaced set of points $\{\theta_j = \frac{\pi j}{N}, j = 0, \dots, N\} \subset [0, \pi]$ to the interval $[-1, 1]$ rather than a mapping from the Chebyshev points to the interval $[-1, 1]$. Also, \tilde{g} will be restricted to be odd with respect to $\frac{\pi}{2}$, i.e., only symmetric distributions of points with respect to the center of the interval $[-1, 1]$ will be considered. This property of \tilde{g} keeps the definitions of the coefficients \hat{a}_k 's as in (7) and (8) unaltered and everything is the same up to the definition of the auxiliary functions $\Psi_{\pm}(\theta) = \beta \tilde{g}(\theta) \pm \theta$. Clearly the spatial resolution will depend on $\|\tilde{g}'\|_{\infty}$, which controls the minimal value of k in the ratio $\beta = \frac{m}{k}$ in order for $\Psi_{\pm}(\theta)$ to inherit the strict monotonicity of the linear term $\pm \theta$, This monotonicity is necessary for the nonexistence of stationary points and the consequential $O(k^{-1})$ rate of decay.

For instance, considering the classical cases of an equi-spaced distribution and the Chebyshev set itself, respectively given by the mappings

$$g_e = 1 - \frac{2\theta}{\pi} \tag{16}$$

and

$$g_c = \cos(\theta), \tag{17}$$

we obtain that $\frac{m}{k} \|\tilde{g}'\|_{\infty} \pm 1 \neq 0$ iff

$$k > \frac{2m}{\pi} = 2N \quad \text{if } \tilde{g} = g_e \tag{18}$$

and

$$k > m = N\pi \quad \text{if } \tilde{g} = g_c. \tag{19}$$

It is very interesting to notice that the resolution condition (19) has been only heuristically demonstrated in the past literature, through the association of the Chebyshev series with Bessel coefficients in (4), although in [9], the sufficiency side of this condition has been shown, i.e., that π points per wavelength is sufficient for resolution. More interesting yet is to observe the sharpness of the results provided by this technique. In equation (9.1.61) in [2], if $k = N\pi$, yields:

$$0 < J_k(k) < \frac{\sqrt{2}}{3^{2/3}\Gamma(2/3)k^{1/3}},$$

showing that (19) is strict. This same result is also analytically obtained through the auxiliary functions $\Psi_{\pm} = \beta\tilde{g}_c \pm \theta$, analogously as in the demonstration of Theorem 1, by noticing that $\Psi'_+(\frac{\pi}{2}) = \Psi''_+(\frac{\pi}{2}) = 0$ and $\Psi'''_+(\frac{\pi}{2}) \neq 0$.

5 Conclusions

In this work we discussed the spatial resolution power of the Kosloff–Tal-Ezer mapping of the Chebyshev points. Using the well-known Method of Stationary Phase we solved the conjecture on the superior spatial resolution of the mapped Chebyshev method with respect to the classical Chebyshev method, shedding more light on what has been a controversial subject in the literature. The techniques developed in this article are useful for the analysis of other mappings. In particular, we were able to demonstrate the classical result on the resolution power of the Chebyshev polynomials without resorting to the usual Fourier-Bessel series convergence.

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